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Local solution for a class of mixed boundary value problems

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Abstract

A local method is developed for solving locally partial differential equations with mixed boundary conditions. The method is based on a heuristic idea, properties of diffusion processes, stopping times and the Itô formula for semimartingales. According to the heuristic idea, the diffusion process used for solving locally a partial differential with mixed boundary conditions is stopped when it reaches a Neumann boundary and then restarted inside the domain of definition of this equation at a point depending on the Neumann conditions. The proposed method is illustrated and its accuracy assessed by two simple numerical examples solving locally mixed boundary value problems in one and two space dimensions.

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1. Introduction

Many problems in science and engineering can be described by deterministic partial differential equations. Generally, these equations do not admit analytical solutions but can be solved numerically [5]. Most available numerical methods, for example, the finite element, boundary element and finite difference methods, are global, that is they determine the solution everywhere or at a large number of points of a set $D \subset \mathbb{R}^d$ in which the solution is defined. Some possible limitations of the traditional computational methods are (1) the computer codes used for solution are relatively complex and may require extensive preprocessing to define a particular problem in a specified format, (2) the numerical algorithms may become unstable in some cases, (3) the errors caused by the discretization of the domain of integration and the numerical integration methods used in analysis cannot be bounded, and (4) the field solution must be calculated even if the solution is needed at a single point or a small collection of points in D [1].

This paper develops an alternative method for solving a class of deterministic partial differential equations with mixed boundary conditions. The method is local, that is it

gives directly the value of the solution for some differential equations at an arbitrary point in D, rather than extracting its value from the field solution, and is based on properties of stochastic processes, Itô's formula for semimartingales and Monte Carlo simulation. Numerical algorithms based on the local solution in this paper are simple to program, always stable, accurate, local and ideal for parallel computation. However, the proposed local solution cannot solve partial differential equations of an arbitrary type.

The class of deterministic partial differential equations that can be solved by the method in this paper has the form

$$\sum_{i=1}^{d} \alpha_i(\boldsymbol{x}) \frac{\partial u(\boldsymbol{x})}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \beta_{ij}(\boldsymbol{x}) \frac{\partial^2 u(\boldsymbol{x})}{\partial x_i \partial x_j} + p(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in D \tag{1}$$

where D is an open subset of $\mathbb{R}^d, d \geqslant 1$, with a compact closure $\bar{D}, \alpha_i, \beta_{ij}$ are real-valued functions defined on D and p denotes a real-valued function defined on D. This class of equations includes many problems in physics and engineering, for example, stationary heterogeneous diffusion equations with drift and solid mechanics problems ([6], chapter 6). It is assumed that p has a continuous extension to $\bar{D}$, and the functions α_i, β_{ij} also have extensions to $\bar{D}$ that are bounded and satisfy the uniform Lipschitz conditions. Two types of boundary value problems are considered for equation (1), Dirichlet and mixed boundary value problems. For Dirichlet problems the value of u is specified at all points of the boundary ∂D of D, that is

$$\lim_{\boldsymbol{y} \to \boldsymbol{x}, \boldsymbol{y} \in D} u(\boldsymbol{y}) = \xi_d(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial D \tag{2}$$

where ξ_d is a known function. Local solutions for many Dirichlet boundary value problems are discussed in [6] (sections 6.2.1 and 6.2.2), and only briefly reviewed in this paper. For mixed problems u is specified only on a subset ∂D_d of ∂D but is not known on $\partial D_n = \partial D \backslash \partial D_d$. The boundary conditions for these problems can be given in the form

$$\begin{aligned} &\lim_{\boldsymbol{y} \to \boldsymbol{x}, \boldsymbol{y} \in D} u(\boldsymbol{y}) = \xi_d(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial D_d \\ &\lim_{\boldsymbol{y} \to \boldsymbol{x}, \boldsymbol{y} \in D} \nabla u(\boldsymbol{y}) \cdot \boldsymbol{c}(\boldsymbol{y}) = \xi_n(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial D_n \end{aligned} \tag{3}$$

where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$ and $\boldsymbol{c}$ and ξ_n are prescribed $\mathbb{R}^d$- and real-valued functions, respectively. We assume that $\boldsymbol{c}$ has a continuous extension to $\bar{D}$, and that with these boundary conditions equation (1) has a solution u with bounded second-order partial derivatives in $\bar{D}$. The proposed local solution for mixed boundary value problems is general for $D \subset \mathbb{R}$ but applies only to problems that admit a regular solution for $D \subset \mathbb{R}^d, d \geqslant 2$. The extension of the local solution to general mixed boundary value problems with $D \subset \mathbb{R}^d, d \geqslant 2$ will be addressed in a later work.

Consider an increasing sequence of smooth open subsets $D_k, k = 1, 2, \ldots$, of D such that $\bar{D}_k \subset D$ and $\cup_{k=1}^{\infty} D_k = D$ ([2], chapter 7). Then $\rho_k = \sup_{\boldsymbol{x} \in \partial D_k} \inf_{\boldsymbol{y} \in \partial D} \|\boldsymbol{x} - \boldsymbol{y}\| \to 0$ as $k \to \infty$. We can talk about Dirichlet and Neumann boundaries for D_k in the following sense. For $\boldsymbol{x} \in \partial D_k$ let

$$\partial D(\boldsymbol{x}) = \left\{ \boldsymbol{y}_0 \in \partial D : \inf_{\boldsymbol{y} \in \partial D} \| \boldsymbol{x} - \boldsymbol{y} \| = \| \boldsymbol{x} - \boldsymbol{y}_0 \| \right\}$$

and define

$$\partial D_{k,d} = \{ \boldsymbol{x} \in \partial D_k : \partial D(\boldsymbol{x}) \cap \partial D_d \neq \emptyset \}$$

and $\partial D_{k,n} = \partial D_k \backslash \partial D_{k,d}$. Intuitively, $\partial D_{k,d}$ and $\partial D_{k,n}$ consist of points of ∂D_k that are closer to ∂D_d than to ∂D_n and vice versa (figure 1). For $\boldsymbol{x} \in \partial D_{k,d}$ let $\boldsymbol{y}_{\boldsymbol{x}}$ be defined as the point

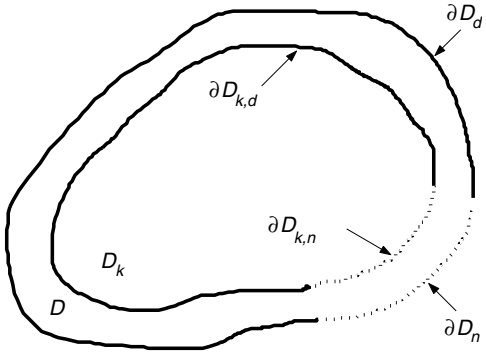

Figure 1. Approximations of D.

in $\partial D(\boldsymbol{x}) \cap \partial D_d$ with the smallest value of ξ_d (break ties in, say, lexicographical order) and, similarly, for $\boldsymbol{x} \in \partial D_{k,n}$ we let $\boldsymbol{y}_{\boldsymbol{x}}$ be the point in $\partial D(\boldsymbol{x}) \cap \partial D_n$ with the smallest value of ξ_n, with similarly broken ties. Note that

$$\begin{aligned} &\lim_{k\to\infty} \sup_{\boldsymbol{x}\in D_{k,d}} |u(\boldsymbol{x}) - \xi_d(\boldsymbol{y}_{\boldsymbol{x}})| = 0 \\ &\lim_{k\to\infty} \sup_{\boldsymbol{x}\in D_{k,n}} |\nabla u(\boldsymbol{x}) \cdot \boldsymbol{c}(\boldsymbol{x}) - \xi_n(\boldsymbol{y}_{\boldsymbol{x}})| = 0. \end{aligned} \tag{4}$$

It is assumed that D_k has only regular points, and that the set of points in ∂D_k where the Dirichlet and Neumann boundaries meet is finite. Finally, we assume that the field $\boldsymbol{c}$ is such that $\boldsymbol{y} + \varepsilon \boldsymbol{c}(\boldsymbol{y}) \in D_k$ for all k large enough, $\varepsilon > 0$ small enough and all $\boldsymbol{y} \in \partial D_{k,n}$. We say that $\boldsymbol{y} \in \partial D_k$ is a regular point with respect to an $\mathbb{R}^d$-valued process $\boldsymbol{X}$ if $P(S = 0) = 1$ when $\boldsymbol{X}(0) = \boldsymbol{y}$, where $S = \inf\{t > 0 : \boldsymbol{X}(t) \notin D_k\}$.

The method for local solutions in the paper can be extended to solve partial differential equations that generalize equation (1). For example, extended versions of equation (1) may include terms of the form qu and $\partial u/\partial t$ ([6], chapter 6).

2. Dirichlet boundary value problems

Let u be the solution of equation (1) with the boundary condition in equation (2). The objective is to find the value of the unknown function u at an arbitrary point $\boldsymbol{x} \in D$, that is to solve locally equation (1) with the boundary condition in equation (2).

Consider an $\mathbb{R}^d$-valued diffusion process defined by the stochastic differential equation

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{a}(\boldsymbol{X}(t))\,\mathrm{d}t + \boldsymbol{b}(\boldsymbol{X}(t))\,\mathrm{d}\boldsymbol{B}(t) \qquad t \geqslant 0 \tag{5}$$

where $a_i(\boldsymbol{x}) = \alpha_i(\boldsymbol{x})$, $i = 1, \ldots, d$, $\boldsymbol{b}$ is such that $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x})^T$ for all $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{B}$ is an $\mathbb{R}^d$-valued Brownian motion process whose coordinates are independent Brownian motions. The solution of equation (5) exists and is unique by the properties of the coefficients α_i, β_{ij} of equation (1). Let

$$T = \inf\{t \geqslant 0 : \boldsymbol{X}(t) \notin D\} \tag{6}$$

be an $\mathcal{F}_t = \sigma(\boldsymbol{B}(s), 0 \leqslant s \leqslant t)$-stopping time giving the first time when $\boldsymbol{X}$ starting at $\boldsymbol{x} \in D$ exits D.

If $E^{\boldsymbol{x}}[T] < \infty$, then the local solution of equation (1) is

$$u(\boldsymbol{x}) = E^{\boldsymbol{x}}[\xi_d(\boldsymbol{X}(T))] + E^{\boldsymbol{x}}\left[\int_0^T p(\boldsymbol{X}(s))\,\mathrm{d}s\right]. \tag{7}$$

The above expectations can be obtained analytically only in some special cases. However, they can be estimated simply by Monte Carlo simulation ([6], section 6.2.1.1).

That equation (7) holds results from the following arguments. Since $\boldsymbol{X}$ is a continuous semimartingale, the Itô formula can be applied to a function $g(\boldsymbol{X}(t))$, $g \in \mathcal{C}^2(\mathbb{R}^d)$, and gives

$$g(\boldsymbol{X}(t)) - g(\boldsymbol{X}(0)) = \int_0^t \sum_{i=1}^d \frac{\partial g(\boldsymbol{X}(s))}{\partial x_i}\,\mathrm{d}X_i(s) + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 g(\boldsymbol{X}(s))}{\partial x_i \partial x_j}\,\mathrm{d}[X_i, X_j](s) \quad (8)$$

where $[X_i, X_j]$ denotes the quadratic covariation process of X_i and X_j so that we have $\mathrm{d}[X_i, X_j](s) = (\boldsymbol{b}(\boldsymbol{X}(s))\boldsymbol{b}(\boldsymbol{X}(s))^T)_{ij}\,\mathrm{d}s$ ([9], theorem 33, p 74). If $\boldsymbol{X}(0) = \boldsymbol{x} \in D$, the expectation of equation (8) becomes

$$E^{\boldsymbol{x}}[g(\boldsymbol{X}(t))] - g(\boldsymbol{x}) = E^{\boldsymbol{x}}\left[\int_0^t \sum_{i=1}^d \frac{\partial g(\boldsymbol{X}(s))}{\partial x_i}\alpha_i(\boldsymbol{X}(s))\,\mathrm{d}s + \frac{1}{2}\int_0^t \sum_{i,j=1}^d \frac{\partial^2 g(\boldsymbol{X}(s))}{\partial x_i \partial x_j}\beta_{ij}(\boldsymbol{X}(s))\,\mathrm{d}s\right] \quad (9)$$

by the properties of $\boldsymbol{X}$. Since stopped semimartingales are semimartingales, equation (9) holds for the stopped process $\boldsymbol{X}^{T_k}(t) = \boldsymbol{X}(t \wedge T_k)$, $t \geqslant 0$, where $t \wedge T_k = \min(t, T_k)$ and

$$T_k = \inf\{t \geqslant 0 : \boldsymbol{X}(t) \notin D_k\} \qquad k = 1, 2, \ldots.$$

Also, the function g in equation (9) can be replaced by u in equation (1) since u has continuous second-order partial derivatives in D_k. These changes implemented in equation (9) yield

$$u(\boldsymbol{x}) = E^{\boldsymbol{x}}[u(\boldsymbol{X}(t \wedge T_k))] + E^{\boldsymbol{x}}\left[\int_0^{t\wedge T_k} p(\boldsymbol{X}(s))\,\mathrm{d}s\right] \quad (10)$$

by the defining equation for u since $\boldsymbol{X}(s) \in D$ for $s < t \wedge T_k$. It remains to show that the limit of equation (10) as $t \to \infty$ and then $k \to \infty$ gives the local solution in equation (7). For a fixed k the sequence $V_k^{(j)} = u(\boldsymbol{X}(j \wedge T_k))$, $j = 1, 2, \ldots$, is bounded by assumption and converges a.s. to $V_k = u(\boldsymbol{X}(T_k))$ as $j \to \infty$. Hence, $E^{\boldsymbol{x}}\left[V_k^{(j)}\right]$ converges to $E^{\boldsymbol{x}}[V_k] = E^{\boldsymbol{x}}[u(\boldsymbol{X}(T_k))]$ as $j \to \infty$ by bounded convergence. Since the function p is assumed to be bounded in $\bar{D}$ and $E^{\boldsymbol{x}}[T] < \infty$, for a fixed k the sequence $W_k^{(j)} = \int_0^{j\wedge T_k} p(\boldsymbol{X}(s))\,\mathrm{d}s$, $k = 1, 2, \ldots$, is uniformly integrable ([8], theorem C.3, p 295) and converges a.s. to $W_k = \int_0^{T_k} p(\boldsymbol{X}(s))\,\mathrm{d}s$ as $j \to \infty$, so that $\lim_{j\to\infty} E^{\boldsymbol{x}}\left[W_k^{(j)}\right] = E^{\boldsymbol{x}}[W_k]$ ([8], theorem C.4, p 296). Therefore, for every k we have

$$u(\boldsymbol{x}) = E^{\boldsymbol{x}}[u(\boldsymbol{X}(T_k))] + E^{\boldsymbol{x}}\left[\int_0^{T_k} p(\boldsymbol{X}(s))\,\mathrm{d}s\right].$$

Since $T_k \uparrow T$ as $k \to \infty$, we can let $k \to \infty$ in the above relation and use the same arguments as above to obtain equation (7).

3. Mixed boundary value problems

Consider equation (1) defined on an open bounded subset D with the mixed boundary conditions in equation (3). The objective is to find the value of $u(\boldsymbol{x})$ at an arbitrary point $\boldsymbol{x} \in D$. A heuristic method for finding the local solution is first presented. It is then shown that the heuristic method provides accurate approximations for the local solution of equation (1).

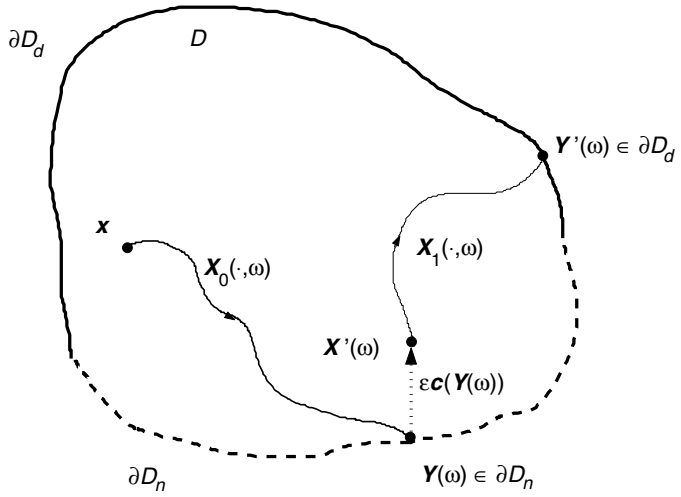

Figure 2. Heuristic method.

3.1. Heuristic method

Let $\boldsymbol{X}$ be the diffusion process in equation (5) with $\boldsymbol{X}(0) = \boldsymbol{x} \in D$. The sample paths of $\boldsymbol{X}$ can be divided into two groups, sample paths that reach ∂D for the first time at points $\boldsymbol{y} \in \partial D_d$ and $\boldsymbol{y} \in \partial D_n$, respectively. The contribution to the local solution of the samples reaching ∂D_d is of the type in equation (7). The formula in equation (7) cannot be used to find the contribution to the local solution of the samples reaching ∂D_n since u is not known on ∂D_n. This contribution can be obtained by the following heuristic algorithm. First, apply the Itô formula between $\boldsymbol{x}$ and $\boldsymbol{y} \in \partial D_n$ to relate the values of u at $\boldsymbol{x} \in D$ and $\boldsymbol{y} \in \partial D_n$. Second, reflect $\boldsymbol{X}$, that is, shift $\boldsymbol{y} \in \partial D_n$ to a point $\boldsymbol{x}' \in D$ by a selected amount in a direction consistent with the Neumann boundary condition at $\boldsymbol{y} \in \partial D_n$. Third, restart $\boldsymbol{X}$ at $\boldsymbol{x}' \in D$ and find the point where the restarted process reaches ∂D for the first time. If this point is on ∂D_d the sample of $\boldsymbol{X}$ is stopped because u is known on ∂D_d. Otherwise, the previous step is repeated till the restarted process reaches ∂D_d.

Figure 2 illustrates the heuristic method. Let $\boldsymbol{X}_0(\cdot, \omega)$ be the diffusion process in equation (5) starting at $\boldsymbol{x} \in D$. Suppose that $\boldsymbol{X}_0$ reaches the boundary of D for the first time at $\boldsymbol{Y}(\omega) \in \partial D_n$. The point $\boldsymbol{Y}(\omega)$ is reflected to $\boldsymbol{X}'(\omega) \in D$ such that $\boldsymbol{X}'(\omega) - \boldsymbol{Y}(\omega) = \varepsilon \boldsymbol{c}(\boldsymbol{Y}(\omega))$, where $\varepsilon > 0$ is a specified constant and $\boldsymbol{c}$ is defined in equation (3). Let $\boldsymbol{X}_1(s, \omega)$, $s \geqslant 0$, be a sample of $\boldsymbol{X}$ in equation (5) starting at $\boldsymbol{X}'(\omega) \in D$. Suppose that $\boldsymbol{X}_1(\cdot, \omega)$ reaches ∂D for the first time at $\boldsymbol{Y}'(\omega) \in \partial D_d$. The relationship between the values of u at $\boldsymbol{x}$ and $\boldsymbol{Y}(\omega)$ is (equation (7))

$$u(\boldsymbol{x}, \omega) = u(\boldsymbol{Y}(\omega)) + \int_0^{S(\omega)} p(\boldsymbol{X}(s, \omega))\, \mathrm{d}s$$

where $S(\omega)$ denotes the time it takes the sample $\boldsymbol{X}$ starting at $\boldsymbol{x}$ to reach $\boldsymbol{Y}(\omega)$. The values of u at $\boldsymbol{Y}(\omega)$ and $\boldsymbol{X}'(\omega)$ can be related by

$$u(\boldsymbol{Y}(\omega)) \simeq u(\boldsymbol{X}'(\omega)) - \varepsilon \nabla u(\boldsymbol{Y}(\omega)) \cdot \boldsymbol{c}(\boldsymbol{Y}(\omega)) = u(\boldsymbol{X}'(\omega)) - \varepsilon \xi_n(\boldsymbol{Y}(\omega))$$

where the first approximate equality is adequate for a small ε and the last equality is the boundary condition in equation (3). The local solution in equation (7) also gives the relationship

$$u(\boldsymbol{X}'(\omega)) = \xi_d(\boldsymbol{Y}'(\omega)) + \int_0^{S_1(\omega)} p(\boldsymbol{X}_1(s,\omega))\,\mathrm{d}s$$

where $\boldsymbol{X}_1$ is defined by equation (5) with the initial value $\boldsymbol{X}'(\omega)$ and $S_1(\omega)$ denotes the time it takes a sample of this process to reach $\boldsymbol{Y}'(\omega)$. The above equations show that the contribution to the local solution of the sample in figure 2 can be approximated by

$$u(\boldsymbol{x},\omega) \simeq \xi_d(\boldsymbol{Y}'(\omega)) - \varepsilon\xi_n(\boldsymbol{Y}(\omega)) + \int_0^{S(\omega)} p(\boldsymbol{X}(s,\omega))\,\mathrm{d}s + \int_0^{S_1(\omega)} p(\boldsymbol{X}_1(s,\omega))\,\mathrm{d}s.$$

The unknown value of $u(\boldsymbol{x})$ can be estimated by the arithmetic average of samples $u(\boldsymbol{x},\omega)$ generated by the Monte Carlo simulation.

3.2. Exact method. Laplace equation for $D \subset \mathbb{R}$

Let u be the solution of the differential equation

$$u''(x) + p(x) = 0 \qquad x \in (0,1) \tag{11}$$

where $p \in \mathcal{C}[0,1]$ and $u'(0), u(1)$ are specified. The objective is to find the local solution for equation (11), that is the value of u at an arbitrary $x \in (0,1)$ satisfying the boundary conditions $\lim_{x\to 0, x\in(0,1)} u'(x) = u'(0)$ and $\lim_{x\to 0, x\in(0,1)} u(x) = u(1)$.

Let $B(t)$, $t \geqslant 0$, be a Brownian motion starting at $B(0) = x \in (0,1)$, and choose a kth approximating domain $D_k = (c_k, d_k)$ of $(0,1)$ such that $x \in D_k$, the differences under the limits in equation (4) are small, and $d_k - c_k > \varepsilon$, for some $\varepsilon \in (0,1)$. Define a sequence of random times $U_1 = \inf\{t \geqslant 0 : B(t) = c_k\}$ and for $n \geqslant 2$, $U_n = \inf\{t \geqslant 0 : B(t) = c_k - (n-1)\varepsilon\}$. The sequence $U_1, U_2, \ldots$ should be denoted by $U_{k,1}, U_{k,2}, \ldots$ since it depends on k, but the index k is suppressed for simplicity. Consider the process $Y_\varepsilon(t) = B(t) + L_\varepsilon(t)$, $t \geqslant 0$, where

$$L_\varepsilon(t) = \varepsilon \sum_{n\geqslant 1} 1(U_n \leqslant t) \tag{12}$$

and let $T_k = \inf\{t \geqslant 0 : Y_\varepsilon(t) = c_k\}$ be the first time when Y_ε reaches the Dirichlet boundary $x = c_k$. Denote by

$$N_\varepsilon^{(k)}(t) = \sup\{n \geqslant 0 : U_n \leqslant t\} \tag{13}$$

the number of jumps of L_ε and Y_ε in $(0,t]$.

Since p and u are bounded in $[0,1]$, the local solution of equation (11) is

$$u(x) = u(d_k) + \frac{1}{2}E^x\left[\int_0^{T_k} p(Y_\varepsilon(s))\,\mathrm{d}s\right] - (u(c_k+\varepsilon) - u(c_k))E^x\left[N_\varepsilon^{(k)}(T_k)\right] \tag{14}$$

and $E^x\left[N_\varepsilon^{(k)}(T)\right] = (d_k - x)/(\varepsilon - c_k)$.

The validity of equation (14) results from the following considerations. First, the random variables T_k and $U_1 \leqslant U_2 \leqslant \cdots$ are $\mathcal{F}_t = \sigma(B(s), 0 \leqslant s \leqslant t)$-stopping times. Second, the process Y_ε is an $\mathcal{F}_t$-semimartingale since B is a martingale with respect to its natural filtration and L_ε is an $\mathcal{F}_t$-adapted process with increasing samples that are right continuous with left limits, so that it is a semimartingale ([9], theorem 1, p 88). The Itô formula for semimartingales

applied to $g(X)$, $g \in \mathcal{C}^2(\mathbb{R})$, gives ([9], theorem 32, p 71)

$$g(Y_\varepsilon(t)) - g(Y_\varepsilon(0)) = \int_{0+}^{t} g'(Y_\varepsilon(s-))\,\mathrm{d}Y_\varepsilon(s) + \frac{1}{2}\int_{0+}^{t} g''(Y_\varepsilon(s-))\,\mathrm{d}[Y_\varepsilon, Y_\varepsilon]^c(s) + \sum_{0<s\leqslant t} [g(Y_\varepsilon(s)) - g(Y_\varepsilon(s-)) - g'(Y_\varepsilon(s-))\Delta Y_\varepsilon(s)]$$

where $[Y_\varepsilon, Y_\varepsilon]^c$ is the continuous part of the quadratic variation of Y_ε so that $\mathrm{d}[Y_\varepsilon, Y_\varepsilon]^c(s) = \mathrm{d}s$ and $\Delta Y_\varepsilon(s) = Y_\varepsilon(s) - Y_\varepsilon(s-)$ denotes the jump of Y_ε at time s. An alternative form of the above equation is

$$g(Y_\varepsilon(t)) - g(Y_\varepsilon(0)) = \int_{0+}^{t} g'(Y_\varepsilon(s-))\,\mathrm{d}B(s) + \frac{1}{2}\int_{0+}^{t} g''(Y_\varepsilon(s-))\,\mathrm{d}s + \sum_{U_n\leqslant t} [g(Y_\varepsilon(U_n)) - g(Y_\varepsilon(U_n-))] \tag{15}$$

where $\sum_{U_n\leqslant t} [g(Y_\varepsilon(U_n)) - g(Y_\varepsilon(U_n-))] = (g(c_k+\varepsilon) - g(c_k))\, N_\varepsilon^{(k)}(t)$ since the jumps of Y_ε have magnitude ε. Since Y_ε is a semimartingale so is the stopped process $Y_\varepsilon^{T_k}(\cdot) = Y_\varepsilon(\cdot \wedge T_k)$ so that t can be replaced by $t \wedge T_k$ in the above equation. Also, since u in equation (11) has a continuous second-order derivative in D, equation (15) holds with u in place of g. These changes implemented in equation (15) give, as long as $x \in D_k$,

$$u(x) = E^x[u(Y_\varepsilon(t \wedge T_k))] + \frac{1}{2}E^x\left[\int_0^{t\wedge T_k} p(Y_\varepsilon(s))\,\mathrm{d}s\right] - (u(c_k+\varepsilon) - u(c_k))E^x\left[N_\varepsilon^{(k)}(t \wedge T_k)\right] \tag{16}$$

by averaging and using the defining equation for u, which holds for points $Y_\varepsilon(s)$ with $s < t \wedge T_k$. The integral $\int_{0+}^{t\wedge T_k} u''(Y_\varepsilon(s-))\,\mathrm{d}s$ is equal a.s. to $\int_0^{t\wedge T_k} u''(Y_\varepsilon(s))\,\mathrm{d}s$ since the set of jumps of Y_ε in $[0, t \wedge T_k]$ has measure 0. Letting, as in the previous section, $t \to \infty$ establishes equation (14).

We now compare the exact solution above with the approximate local solution by the heuristic method in the previous section. Note that because of homogeneity of the Brownian motion, we could define the times U_n through the original process B instead of the process Y_ε. We will not be able to do this in general, as will be seen in the next sections. The approximate local solution is

$$\tilde{u}(x) = u(1) + \frac{1}{2}E^x\left[\int_0^T p(Y_\varepsilon(s))\,\mathrm{d}s\right] - \varepsilon u'(0)E^x[N_\varepsilon(T)] \tag{17}$$

so that

$$\begin{aligned}|\tilde{u}(x) - u(x)| \leqslant{}& |u(d_k) - u(1)| + \frac{1}{2}\left|E^x\left[\int_{T_k}^T p(Y_\varepsilon(s))\,\mathrm{d}s\right]\right| \\ &+ |u(c_k+\varepsilon) - u(c_k) - \varepsilon u'(c_k)|E^x\left[N_\varepsilon^{(k)}(T_k)\right] \\ &+ \varepsilon\left|u'(0)E^x[N_\varepsilon(T)] - u'(c_k)E^x\left[N_\varepsilon^{(k)}(T_k)\right]\right|\end{aligned} \tag{18}$$

where T and $N_\varepsilon(T)$ are defined as T_k and $N_\varepsilon^{(k)}(T)$, respectively, but with interval $(0, 1)$ replacing (c_k, d_k). Choosing k so large that the first, second and fourth terms in the right-hand side above have the order of magnitude ε, and using the fact that

$$|u(c_k+\varepsilon) - u(c_k) - \varepsilon u'(c_k)|E^x\left[N_\varepsilon^{(k)}(T_k)\right] \sim c\varepsilon^2$$

for some constant c, we conclude that $|\tilde{u}(x) - u(x)|$ is of order ε.

Figure 3 shows the exact solution of equation (11) with $p(x) = 1$, $u'(0) = 1$ and $u(1) = 1$, and the approximate solution $\tilde{u}$ in equation (17) for $\varepsilon = 0.1$. The expectations in this equation

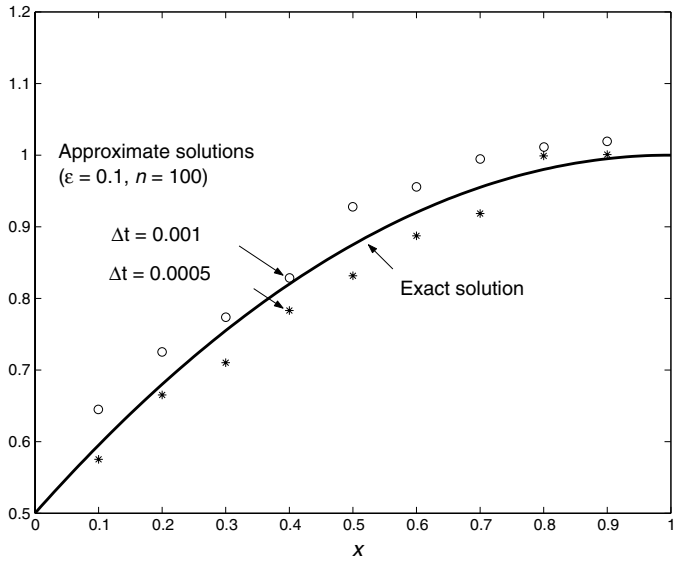

Figure 3. Exact and approximate solutions.

have been estimated from $n = 100$ independent samples of Y_ε generated with time steps $\Delta t = 0.001$ and $\Delta t = 0.0005$. The resulting local solution is in error by less than 10%. The performance of the local solution can be improved by increasing the sample size and/or decreasing the time step. For example, if the sample size is increased to $n = 500$, the error of the local solution is less that -4.5% and 0.0356% for $\Delta t = 0.001$ and $\Delta t = 0.0005$, respectively.

The local solution of equation (11) in equation (14) can be expressed in another way. For example, let

$$Z(t) = |B(t)| = \hat{B}(t) + L(t) \qquad t \geqslant 0 \tag{19}$$

be a Brownian motion reflected at zero, where $L(t) = \lim_{\varepsilon \downarrow 0} 1/(2\varepsilon) \int_0^t 1(B(s) \in (-\varepsilon, \varepsilon))\, \mathrm{d}s$ is the local time process and $\hat{B}$ denotes a Brownian motion ([3], theorem 7.6, p 150). The Itô formula applied to $u(Z(t))$ with $Z(0) = x$ gives

$$u(x) = u(1) + \frac{1}{2} E^x \left[\int_0^{T^*} p(Z(s))\, \mathrm{d}s \right] - E^x \left[\int_0^{T^*} u'(Z(s))\, \mathrm{d}L(s) \right] \tag{20}$$

where $T^* = \inf\{t \geqslant 0 : Z(t) = 1\}$.

The local solutions in equations (14) and (20) are exact so that they must coincide since equation (11) has a unique solution. However, equation (14) cannot be used to calculate $u(x), x \in (0, 1)$, since it depends on the unknown values $u(0)$ and $u(\varepsilon)$ of u. In contrast, the local solution in equation (20) can be used to calculate $u(x)$ at any point $x \in (0, 1)$. Unfortunately, the approach used to establish equation (19) cannot be extended simply to solve locally partial differential equations of the type in equation (1) for arbitrary sets $D \subset \mathbb{R}^d, d > 1$. Practical results can only be found in some special cases ([3], chapter 8). Therefore, approximations of the type in equation (17) need to be developed for solving equation (1) locally. The following sections consider differential equations more general than equation (11), develop approximate local solutions for these equations and establish the accuracy of these resulting approximate local solutions.

3.3. Exact method. General equation for $D \subset \mathbb{R}$

Suppose that u is the solution of the differential equation

$$a(x)u'(x) + \tfrac{1}{2}b(x)^2u''(x) + p(x) = 0 \qquad x \in (0, 1) \tag{21}$$

where $p \in \mathcal{C}[0, 1]$, a, b are bounded functions in $[0, 1]$ satisfying the uniform Lipschitz conditions. We assume that the function b is bounded away from zero in a neighbourhood of zero, which implies that $u''(x)$ is bounded in a neighbourhood of zero. The objective is to find the local solution of equation (21) for the boundary conditions $\lim_{x\to 0, x\in(0,1)} u'(x) = u'(0)$ and $\lim_{x\to 0, x\in(0,1)} u(x) = u(1)$, where $u'(0)$ and $u(1)$ are specified.

Let X be a diffusion process defined by the stochastic differential equation

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}B(t) \qquad t \geqslant 0. \tag{22}$$

Take a sub-domain D_k as before (here, as in the previous section, it can be just a subinterval (c_k, d_k) of $(0, 1)$ with the same requirements) and let $U_1 = \inf\{t \geqslant 0 : X(t) = c_k\}$ be the first time when X reaches the boundary $x = c_k$. If $U_1 = \infty$, define $U_n = \infty$, $n \geqslant 2$. Otherwise, $U_2 \geqslant U_1$ denotes the first time when X restarted at $X(U_1) = \varepsilon$ reaches $x = 0$, where $\varepsilon \in (0, 1)$ is arbitrary. The random times $U_3 \leqslant U_4 \leqslant \cdots$ are defined in the same way. It is assumed that the point d_k is an accessible boundary for X.

Consider also a process Y_ε defined by the differential equation

$$\mathrm{d}Y_\varepsilon(t) = a(Y_\varepsilon(t-))\,\mathrm{d}t + b(Y_\varepsilon(t-))\,\mathrm{d}B(t) + \mathrm{d}L_\varepsilon(t) \qquad t \geqslant 0 \tag{23}$$

where B is the Brownian motion in equation (22) and

$$L_\varepsilon(t) = \varepsilon \sum_{n\geqslant 1} 1(U_n \leqslant t). \tag{24}$$

Let $N_\varepsilon^{(k)}(t) = \sup\{n \geqslant 0 : U_n \leqslant t\}$ be the number of jumps of Y_ε in $(0, t]$ and denote by

$$T_k = \inf\{t \geqslant 0 : Y_\varepsilon(t) = d_k\} \tag{25}$$

the first time when Y_ε reaches the Dirichlet boundary $x = d_k$. Figure 4 shows hypothetical samples of the processes Y_ε and L_ε defined by equations (23) and (24).

If $E^x[T_k]$ is finite (in particular, the probability that Y_ε hits d_k before c_k must be strictly positive), then the local solution for equation (21) is

$$u(x) = u(d_k) + E^x\left[\int_0^{T_k} p(Y_\varepsilon(s))\,\mathrm{d}s\right] - (u(c_k + \varepsilon) - u(c_k))E^x\big[N_\varepsilon^{(k)}(T_k)\big]. \tag{26}$$

The following arguments, similar to what we used above, prove equation (26). First, note that the random times $U_1, U_2, \ldots$ are $\mathcal{F}_t = \sigma(B(s), 0 \leqslant s \leqslant t)$-stopping times. That U_1 is an $\mathcal{F}_t$-stopping time follow from its definition. Let $X_1(s)$, $s \geqslant 0$, be a diffusion process defined by

$$\begin{cases} \mathrm{d}X_1(s) = a(X_1(s))\,\mathrm{d}s + b(X_1(s))\,\mathrm{d}B_1(s) & s \geqslant 0 \\ X_1(0) = c_k + \varepsilon, \end{cases}$$

where $B_1(t) = B(U_1 + t) - B(U_1)$, $t \geqslant 0$. By the strong Markov property of the Brownian motion, B_1 is a Brownian motion with respect to the filtration $\mathcal{H}_t = \mathcal{F}_{U_1+t}$, $t \geqslant 0$ ([4], proposition 1.4, p 52). Therefore, $X_1(s)$, $s \geqslant 0$, is a diffusion process adapted to $\mathcal{H}_t$. Denote by $U_2' = \inf\{s \geqslant 0 : X_1(s) = 0\}$ the first time X_1 reaches the boundary $x = 0$. Then U_2' is an $\mathcal{H}_t$-stopping time, so that $U_2 = U_1 + U_2'$ is an $\mathcal{F}_t$-stopping time. Similar arguments can be used to show that $U_3, U_4, \ldots$ are $\mathcal{F}_t$-stopping times. Second, the process L_ε in equation (24) is a semimartingale with respect to the filtration $\mathcal{F}_t$ since it has increasing samples that are right continuous with left limits and $0 \leqslant U_1 \leqslant U_2 \leqslant \cdots$ are $\mathcal{F}_t$-stopping times. Consequently, the

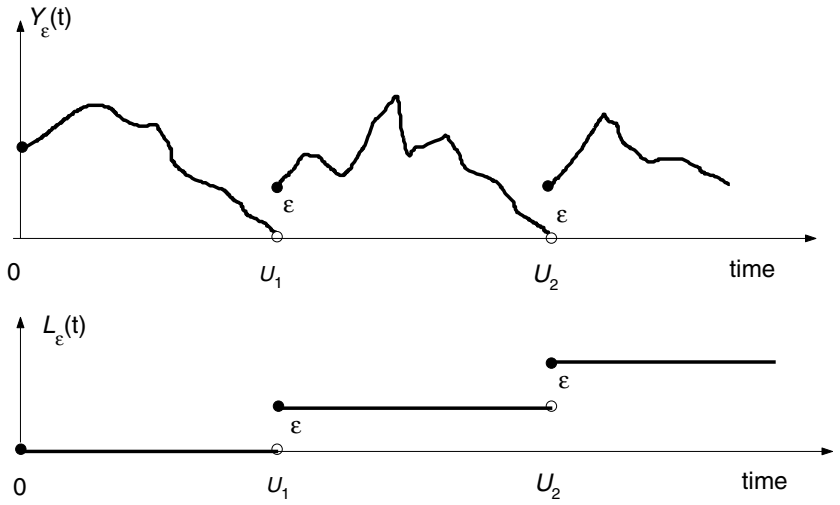

Figure 4. Processes Y_ε and L_ε.

process Y_ε in equation (23) is also a $\mathcal{F}_t$-semimartingale ([9], section V.3). The samples of Y_ε have jumps of magnitude ε at each time U_n. The Itô formula for semimartingales applied to $g(Y_\varepsilon(t))$, $g \in \mathcal{C}^2(\mathbb{R})$, gives

$$g(Y_\varepsilon(t)) - g(Y_\varepsilon(0)) = \int_{0+}^{t} g'(Y_\varepsilon(s-))\,\mathrm{d}Y_\varepsilon(s) + \frac{1}{2}\int_{0+}^{t} g''(Y_\varepsilon(s-))\,\mathrm{d}[Y_\varepsilon, Y_\varepsilon]^c(s) + \sum_{0<s\leqslant t} [g(Y_\varepsilon(s)) - g(Y_\varepsilon(s-)) - g'(Y_\varepsilon(s-))\Delta g(Y_\varepsilon(s))]$$

or

$$g(Y_\varepsilon(t)) - g(Y_\varepsilon(0)) = \int_{0+}^{t} g'(Y_\varepsilon(s-))[a(Y_\varepsilon(s-))\,\mathrm{d}s + b(Y_\varepsilon(s-))\,\mathrm{d}B(s)] + \frac{1}{2}\int_{0+}^{t} g''(Y_\varepsilon(s-))b(Y_\varepsilon(s-))^2\,\mathrm{d}s + \sum_{U_n\leqslant t} [g(Y_\varepsilon(U_n)) - g(Y_\varepsilon(U_n-))] \tag{27}$$

because $\int_{0+}^{t} g'(Y_\varepsilon(s-))\,\mathrm{d}L_\varepsilon(s) = \sum_{0<s\leqslant t} g'(Y_\varepsilon(s-))\Delta g(Y_\varepsilon(s))$ and the continuous part of the quadratic variation of Y_ε is $\mathrm{d}[Y_\varepsilon, Y_\varepsilon]^c(s) = b(Y_\varepsilon(s-))^2\,\mathrm{d}s$. Since stopped semimartingales are semimartingales, equation (27) holds if t is replaced by $t \wedge T_k$. Also, the function g can be replaced by u since this function has a continuous second-order derivative inside D. The expectation of equation (27) with $t \wedge T_k$ and u in place of t and g, respectively, is

$$\begin{aligned} E^x[u(Y_\varepsilon(t \wedge T_k))] - u(x) &= -E^x\left[\int_0^{t\wedge T_k} p(Y_\varepsilon(s))\,\mathrm{d}s\right] + E^x\left[\sum_{U_n\leqslant t\wedge T_k} (u(Y_\varepsilon(U_n)) - u(Y_\varepsilon(U_n-)))\right] \\ &= -E^x\left[\int_0^{t\wedge T_k} p(Y_\varepsilon(s))\,\mathrm{d}s\right] + \varepsilon(u(c_k+\varepsilon) - u(c_k))E^x\left[N_\varepsilon^{(k)}(t \wedge T_k)\right] \end{aligned} \tag{28}$$

by using equation (21) since $Y_\varepsilon(s)$ is in $(0, 1)$ for $s < t \wedge T_k$ and the arguments applied to write equation (16). Third, the limit of equation (28) yields the local solution in equation (26) by considerations as in the previous section.

The local solution by the heuristic method,

$$\tilde{u}(x) = u(1) + E^x\left[\int_{0+}^{T} p(Y_\varepsilon(s-))\,\mathrm{d}s\right] - \varepsilon u'(0)E^x[N_\varepsilon(T)] \tag{29}$$

differs from the solution in equation (26) by

$$\begin{aligned}|\tilde{u}(x) - u(x)| \leqslant{}& |u(d_k) - u(1)| + \frac{1}{2}\left|E^x\left[\int_{T_k}^{T} p(Y_\varepsilon(s))\,\mathrm{d}s\right]\right| \\ &+ |u(c_k+\varepsilon) - u(c_k) - \varepsilon u'(c_k)|E^x[N_\varepsilon(T_k)] \\ &+ \varepsilon|u'(0)E^x[N_\varepsilon(T)] - u'(c_k)E^x[N_\varepsilon(T_k)]|\end{aligned} \tag{30}$$

and with the same choice of k as before we see that

$$|\tilde{u}(x) - u(x)| \leqslant \text{const}\ \varepsilon + |u(c_k+\varepsilon) - u(c_k) - \varepsilon u'(c_k)|E^x\big[N_\varepsilon^{(k)}(T)\big]. \tag{31}$$

For notational simplicity we derive a bound on $E^x[N_\varepsilon(T)]$ instead. Let $q(\xi)$ denote the probability that the diffusion process X in equation (22) starting at $\xi \in (0,1)$ reaches the boundary $x = 1$ prior to $x = 0$. Then $N_\varepsilon(T) = 0, 1, \ldots, m, \ldots$ with the probabilities $q(x), (1-q(x))q(\varepsilon), \ldots, (1-q(x))(1-q(\varepsilon))^{m-1}q(\varepsilon), \ldots$, respectively, so that

$$E^x[N_\varepsilon(T)] = (1-q(x))\sum_{m=1}^{\infty} m(1-q(\varepsilon))^{m-1}q(\varepsilon) = \frac{1-q(x)}{q(\varepsilon)}. \tag{32}$$

The probability $q(x)$ satisfies the ordinary differential equation

$$a(x)q'(x) + \tfrac{1}{2}b(x)^2q''(x) = 0 \tag{33}$$

with the boundary conditions $q(0) = 0$ and $q(1) = 1$ ([7], section 15.3), so that

$$q(x) = \alpha\int^x \mathrm{e}^{-\rho(y)}\,\mathrm{d}y + \beta = \alpha\varphi(x) + \beta \tag{34}$$

where $\rho(y) = \int^y [2a(\sigma)/b(\sigma)^2]\,\mathrm{d}\sigma$ and α, β are some constants. Since q must satisfy the boundary conditions $q(0) = 0$ and $q(1) = 1$, we have

$$q(x) = \frac{\varphi(x) - \varphi(0)}{\varphi(1) - \varphi(0)}.$$

For example, this probability is $q(\xi) = \big(1 - \mathrm{e}^{-2c_1\xi/c_2^2}\big)\big/\big(1 - \mathrm{e}^{-2c_1/c_2^2}\big)$ and $q(\xi) = \xi^{-2c_1/c_2^2+1}$ for u defined by equation (21) with $a(x) = c_1, b(x) = c_2$ and $a(x) = c_1x, b(x) = c_2x$, respectively, where c_1, c_2 are some constants. For a finite limit $\lim_{y\to 0}\rho(y)$, if $\varepsilon > 0$ is small, the probability $q(\varepsilon)$ can be approximated by $\varepsilon\exp(-\rho\varepsilon)/(\varphi(1) - \varphi(0))$ so that $E^x[N_\varepsilon(T)] \sim O(\varepsilon^{-1})$ (equation (32)). As long as u has a bounded second derivative in D, the difference between the exact and the heuristic local solutions is of order ε as $\varepsilon \downarrow 0$ (equation (31)), and the first (but not the second) example above has this property.

3.4. Exact method. General equation for $D \subset \mathbb{R}^d$

Consider the solution u of equation (1) in D_k for some $k \geqslant 1$ (figure 1) with the boundary conditions in equation (4). Let $\boldsymbol{X}$ be the $\mathbb{R}^d$-valued diffusion process in equation (5) with the initial condition $\boldsymbol{X}(0) = \boldsymbol{x} \in D_k$, and let $0 \leqslant U_1 \leqslant U_2 \leqslant \cdots$ be a sequence of random times defined as follows. $U_1 = \inf\{t \geqslant 0 : \boldsymbol{X}(t) \in \partial D_{k,n}\}$ is the first time when $\boldsymbol{X}$ reaches $\partial D_{k,n}$. If $U_1 = \infty$, all the times $U_n, n \geqslant 2$, are also taken to be infinity. Otherwise, $U_2 \geqslant U_1$ is the first time when $\boldsymbol{X}$ restarted at $\boldsymbol{X}(U_1) + \varepsilon\boldsymbol{c}(\boldsymbol{X}(U_1)) \in D_k$ reaches $\partial D_{k,n}$. The random times

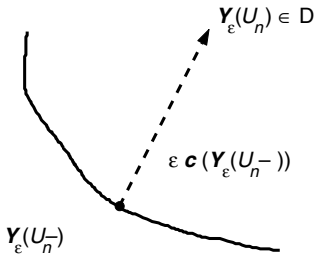

Figure 5. The jumps of $\boldsymbol{Y}_\varepsilon$.

$U_3, U_4, \ldots$ are defined in the same way. Consider also the $\mathbb{R}^d$-valued process $\boldsymbol{Y}_\varepsilon(t), t \geqslant 0$, defined by the differential equation

$$\mathrm{d}\boldsymbol{Y}_\varepsilon(t) = \boldsymbol{a}(\boldsymbol{Y}_\varepsilon(t-))\,\mathrm{d}t + \boldsymbol{b}(\boldsymbol{Y}_\varepsilon(t-))\,\mathrm{d}\boldsymbol{B}(t) + \mathrm{d}\boldsymbol{L}_\varepsilon(t) \qquad t \geqslant 0 \tag{35}$$

with the initial condition $\boldsymbol{Y}_\varepsilon(0) = \boldsymbol{x} \in D$, where

$$\boldsymbol{L}_\varepsilon(t) = \varepsilon \sum_{n \geqslant 1} 1(U_n \leqslant t)\, \boldsymbol{c}(\boldsymbol{Y}_\varepsilon(U_n-)). \tag{36}$$

Figure 5 illustrates the jump of $\boldsymbol{Y}_\varepsilon$ in equation (35) at U_n. Let $N_\varepsilon^{(k)}(t) = \sup\{n \geqslant 0 : U_n \leqslant t\}$ be the number of jumps of $\boldsymbol{Y}_\varepsilon$ in $(0, t]$ and denote by

$$T_k = \inf\{t \geqslant 0 : \boldsymbol{Y}_\varepsilon(t) \in \partial D_{k,d}\} \tag{37}$$

the first time when $\boldsymbol{Y}_\varepsilon$ reaches $\partial D_{k,d}$.

If the expectations $E^{\boldsymbol{x}}[T_k]$ and $E^{\boldsymbol{x}}\big[N_\varepsilon^{(k)}(T_k)\big]$ are finite, then the local solution for equation (1) is

$$u(\boldsymbol{x}) = E^{\boldsymbol{x}}[u(\boldsymbol{Y}_\varepsilon(T_k))] + E^{\boldsymbol{x}}\left[\int_0^{T_k} p(\boldsymbol{Y}_\varepsilon(s))\,\mathrm{d}s\right] - E^{\boldsymbol{x}}\left[\sum_{U_n \leqslant T_k} (u(\boldsymbol{Y}_\varepsilon(U_n) - u(\boldsymbol{Y}_\varepsilon(U_n-))\right]. \tag{38}$$

Arguments as in the previous sections can be used to prove equation (38). First, note that U_1 is an $\mathcal{F}_t = \sigma(\boldsymbol{B}(s), 0 \leqslant s \leqslant t)$-stopping time by its definition. If U_1 is finite, $\boldsymbol{B}_1(s) = \boldsymbol{B}(U_1 + s) - \boldsymbol{B}(U_1)$, $s \geqslant 0$, is a Brownian motion with respect to the filtration $\mathcal{F}_{U_1+t}, t \geqslant 0$, by the strong Markov property of $\boldsymbol{B}$. Hence, $\boldsymbol{X}_1$ defined by

$$\begin{cases} \mathrm{d}\boldsymbol{X}_1(s) = \boldsymbol{a}(\boldsymbol{X}_1(s))\,\mathrm{d}s + \boldsymbol{b}(\boldsymbol{X}_1(s))\,\mathrm{d}\boldsymbol{B}_1(s) & s \geqslant 0 \\ \boldsymbol{X}_1(0) = \boldsymbol{X}(U_1) + \varepsilon \boldsymbol{c}(\boldsymbol{X}(U_1)) \end{cases}$$

is a diffusion process with respect to the same filtration. Defining the random time $U_2' = \inf\{s \geqslant 0 : \boldsymbol{X}_1(s) \in \partial D_{k,n}\}$ we conclude, as above, that $U_2 = U_1 + U_2'$ is an $\mathcal{F}_t$-stopping time. The subsequent times $U_n, n \geqslant 3$, are defined in the same way, and are $\mathcal{F}_t$-stopping time. Similarly, the random variable T_k in equation (37) is an $\mathcal{F}_t$-stopping time. Second, the process $\boldsymbol{L}_\varepsilon$ in equation (36) is an $\mathcal{F}_t$-semimartingale since it is $\mathcal{F}_t$-adapted, has right continuous samples with left limits, and is of bounded variation on compacts ([9], theorem 7, p 47). Hence, $\boldsymbol{Y}_\varepsilon$ in equation (35) is also an $\mathcal{F}_t$-semimartingale ([9], section V.3). The multidimensional Itô formula ([9], theorem 33, p 74) applied to $g(\boldsymbol{Y}_\varepsilon(t))$, $g \in \mathcal{C}^2(\mathbb{R}^d)$,

gives

$$E^{\boldsymbol{x}}[g(\boldsymbol{Y}_\varepsilon(t))] - g(\boldsymbol{x}) = E^{\boldsymbol{x}}\left[\int_{0+}^{t}\sum_{i=1}^{d}\frac{\partial g(\boldsymbol{Y}_\varepsilon(s-))}{\partial y_i}a_i(\boldsymbol{Y}_\varepsilon(s-))\,\mathrm{d}s\right.$$
$$\left.+\frac{1}{2}\int_{0+}^{t}\sum_{i,j=1}^{d}\frac{\partial^2 g(\boldsymbol{Y}_\varepsilon(s-))}{\partial y_i\,\partial y_j}(\boldsymbol{b}(\boldsymbol{Y}_\varepsilon(s-))\boldsymbol{b}(\boldsymbol{Y}_\varepsilon(s-))^T)_{ij}\,\mathrm{d}s\right]$$
$$+E^{\boldsymbol{x}}\left[\sum_{U_n\leqslant t}(g(\boldsymbol{Y}_\varepsilon(U_n)) - g(\boldsymbol{Y}_\varepsilon(U_n-)))\right]$$

after some calculations and averaging. The above equation also holds if t and g are replaced by $t\wedge T_k$ and the solution u of equation (1), respectively. These modifications and the defining equation for u give

$$u(\boldsymbol{x}) = E^{\boldsymbol{x}}[u(\boldsymbol{Y}_\varepsilon(t\wedge T_k))] + E^{\boldsymbol{x}}\left[\int_{0+}^{t\wedge T_k} p(\boldsymbol{Y}_\varepsilon(s-))\,\mathrm{d}s\right]$$
$$-E^{\boldsymbol{x}}\left[\sum_{U_n\leqslant t\wedge T_k}(u(\boldsymbol{Y}_\varepsilon(U_n)) - u(\boldsymbol{Y}_\varepsilon(U_n-)))\right].$$

Third, it remains to show that the limit of the above equation as $t\to\infty$ yields equation (38). The first two terms on the right-hand side of the above equation converge to the corresponding terms in equation (38) by bounded convergence and properties of uniformly integrable random variables. Since $E^{\boldsymbol{x}}\big[N_\varepsilon^{(k)}(T_k)\big]$ is finite, then the last term on the right side of the above equation converges to $E^{\boldsymbol{x}}\big[\sum_{U_n\leqslant T_k}(u(\boldsymbol{Y}_\varepsilon(U_n))-u(\boldsymbol{Y}_\varepsilon(U_n-)))\big]$ as $t\to\infty$ by dominated convergence.

The heuristic solution $\tilde{u}(\boldsymbol{x})$ of section 3 applied to the approximating domain D_k is

$$\tilde{u}(\boldsymbol{x}) = E^{\boldsymbol{x}}\left[\xi_d\big(\boldsymbol{y}_{\boldsymbol{Y}_\varepsilon(T_k)}\big)\right] + E^{\boldsymbol{x}}\left[\int_0^{T_k} p(\boldsymbol{Y}_\varepsilon(s))\,\mathrm{d}s\right]$$
$$-\varepsilon E^{\boldsymbol{x}}\left[\sum_{U_n\leqslant T_k}\nabla\left(\xi_n(\boldsymbol{y}_{\boldsymbol{Y}_\varepsilon(U_n-)}\right)\cdot\boldsymbol{c}(\boldsymbol{Y}_\varepsilon(U_n-)),\right] \tag{39}$$

where $\boldsymbol{y}_{\boldsymbol{Y}_\varepsilon(T_k)}$ has the same meaning as $\boldsymbol{y}_{\boldsymbol{x}}$ in section 1.

If the function u has bounded second derivatives in the domain D, then the same argument as before tells us that for k large enough the difference between the exact and the heuristic local solution $\tilde{u}(\boldsymbol{x})$ in equation (39) and equation (38) is

$$|\tilde{u}(\boldsymbol{x}) - u(\boldsymbol{x})| \leqslant \zeta\varepsilon^2 E^{\boldsymbol{x}}\big[N_\varepsilon^{(k)}(T_k)\big] \tag{40}$$

where $\zeta > 0$ is a finite constant.

The solution of equation (1) in D_k with the boundary conditions in equation (4) rather than in D with the conditions in equation (3) adds to the error of the proposed solution. It is expected that the resulting error is small for sufficiently large values of k. The magnitude of this error is not examined in the paper. Similar approximations are common, for example, in the finite element method, in which D is approximated by a collection of non-overlapping sets with specified geometry, called finite elements. Generally, the union of the finite elements used to represent D does not coincide with D.

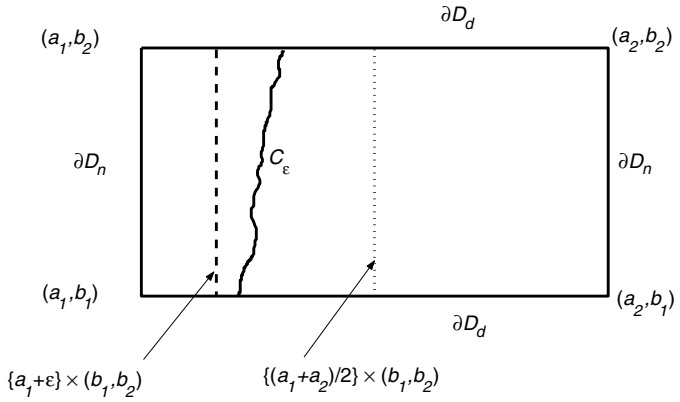

Figure 6. A rectangular domain D for equation (1) with $d = 2$.

3.5. Example: $D = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$

It is difficult to obtain the expression or even the order of $E^{\boldsymbol{x}}\left[N_\varepsilon^{(k)}(T_k)\right]$ in equation (40) for a general setting. However, interesting results can be obtained in some special cases. For example, consider a rectangular domain $D = (a_1, b_1) \times (a_2, b_2)$, $0 \leqslant a_1 < b_1, 0 \leqslant a_2 < b_2$, in $\mathbb{R}^2$ whose horizontal and vertical sides are Dirichlet and Neumann boundaries, respectively. To simplify notation, we use D in our discussion rather than a subset D_k of D (equation (4)). It is shown that for $\varepsilon > 0$ the process $\boldsymbol{Y}_\varepsilon$ reaches a Dirichlet boundary of D with a sufficiently large probability so that $E^{\boldsymbol{x}}[N_\varepsilon(T)]$ is at most of order ε^{-1}. Let $C_\varepsilon \subset D$ be a simple curve connecting the horizontal boundaries of D such that the segment $\{a_1 + \varepsilon\} \times (b_1, b_2)$ is left of C_ε (figure 6). Let $q(\boldsymbol{x})$ be the probability that $\boldsymbol{X}$ starting at $\boldsymbol{x} \in (a_1, a_2) \times (b_1, b_2)$ hits the Dirichlet boundary $(a_1, a_2) \times \{b_1, b_2\}$ before reaching the Neumann boundary $\{a_1, a_2\} \times (b_1, b_2)$. The process $Z(t) = \exp(hX_1(t))$, $t \geqslant 0$, is an $\mathcal{F}_t$-semimartingale so that

$$\begin{aligned} \mathrm{d}Z(t) &= h\,\mathrm{e}^{hX_1(t)}\,\mathrm{d}X_1(t) + \frac{h^2}{2}\,\mathrm{e}^{hX_1(t)}\,\mathrm{d}[X_1, X_1](t) \\ &= m(\boldsymbol{X}(t))\,\mathrm{d}t + h\,\mathrm{e}^{hX_1(t)}(b_{11}(\boldsymbol{X}(t))\,\mathrm{d}B_1(t) + b_{12}(\boldsymbol{X}(t))\,\mathrm{d}B_2(t)) \end{aligned} \tag{41}$$

by the Itô formula, where

$$m(\boldsymbol{x}) = h\,\mathrm{e}^{hx_1}\left(a_1(\boldsymbol{x}) + \frac{h}{2}(b_{11}(\boldsymbol{x})^2 + b_{12}(\boldsymbol{x})^2)\right). \tag{42}$$

Assume that

(1) $\inf_{\boldsymbol{x}\in D}(b_{21}(\boldsymbol{x})^2 + b_{22}(\boldsymbol{x})^2) > 0$ so that $\theta = \inf_{\boldsymbol{x}\in\{(a_1+a_2)/2\}\times(b_1,b_2)} q(\boldsymbol{x}) > 0$, that is there is diffusion in the x_2-direction at any point $\boldsymbol{x} \in D$ and

(2) $\gamma = \inf_{\boldsymbol{x}\in D}(b_{11}(\boldsymbol{x})^2 + b_{12}(\boldsymbol{x})^2) > 0$ so that $m(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in D$ if $h \geqslant \max(0, -2\mu/\gamma)$, where $\mu = \inf_{\boldsymbol{x}\in D} a_1(\boldsymbol{x})$.

Under the latter assumption $Z(t)$, $t \geqslant 0$, is an $\mathcal{F}_t$-submartingale since it is $\mathcal{F}_t = \sigma(\boldsymbol{B}(s), 0 \leqslant s \leqslant t)$-adapted, has a finite expectation for each $t \geqslant 0$, and

$$\begin{aligned} E[\mathrm{d}Z(t) \mid \mathcal{F}_t] &= m(\boldsymbol{X}(t))\,\mathrm{d}t + hZ(t)(b_{11}(\boldsymbol{X}(t))E[\mathrm{d}B_1(t)] + b_{12}(\boldsymbol{X}(t))E[\mathrm{d}B_2(t)]) \\ &= m(\boldsymbol{X}(t))\,\mathrm{d}t \geqslant 0. \end{aligned}$$

Suppose that $\boldsymbol{X}$ starts at a point $\boldsymbol{x} = (x_1, x_2) \in D$ and let $\hat{T}$ be an arbitrary $\mathcal{F}_t$-stopping time, such that $Z(t \wedge T)$, $t \geqslant 0$, is uniformly integrable. Then

$$\mathrm{e}^{hX_1(0)} = \mathrm{e}^{hx_1} = Z(0) \leqslant E^{\boldsymbol{x}}[Z(\hat{T})] \tag{43}$$

by the optional stopping theorem ([4], theorem 2.13, p 61). For

$$\hat{T} = \inf\{t \geqslant 0 : X_1(t) = a_1 \text{ or } X_1(t) = (a_1 + a_2)/2\} \tag{44}$$

and $\boldsymbol{X}(0) = \boldsymbol{x} \in C_\varepsilon$ we have (equation (43))

$$\mathrm{e}^{h(a_1+\varepsilon)} \leqslant \mathrm{e}^{hX_1(0)} = \mathrm{e}^{hx_1} = Z(0) \leqslant E^{\boldsymbol{x}}[Z(\hat{T})].$$

Let $q^*(\boldsymbol{x})$ denote the probability that $\boldsymbol{X}$ starting at $\boldsymbol{x} \in (a_1, (a_1 + a_2)/2) \times (b_1, b_2)$ reaches the vertical line $\{(a_1 + a_2)/2\} \times \mathbb{R}$ dividing D in two halves prior to $\{a_1\} \times \mathbb{R}$. The expectation,

$$E^{\boldsymbol{x}}[Z(\hat{T})] = \mathrm{e}^{h(a_1+a_2)/2} q^*(\boldsymbol{x}) + \mathrm{e}^{ha_1}(1 - q^*(\boldsymbol{x}))$$

and the above equation give $\mathrm{e}^{h(a_1+\varepsilon)} \leqslant Z(0) \leqslant \mathrm{e}^{h(a_1+a_2)/2} q^*(\boldsymbol{x}) + \mathrm{e}^{ha_1}(1 - q^*(\boldsymbol{x}))$ so that

$$q^*(\boldsymbol{x}) \geqslant \frac{\mathrm{e}^{h(a_1+\varepsilon)} - \mathrm{e}^{ha_1}}{\mathrm{e}^{h(a_1+a_2)/2} - \mathrm{e}^{ha_1}} \geqslant c_0 \varepsilon \tag{45}$$

where $c_0 > 0$ is a constant. Since $q(\boldsymbol{x}) \geqslant q^*(\boldsymbol{x})\theta$, we have $q(\boldsymbol{x}) \geqslant c_0 \varepsilon \theta$ for any $\boldsymbol{x} \in C_\varepsilon$. An inductive argument shows that $P(T_n < T) \leqslant (1 - c_0\varepsilon\theta)^{n-1}$ for any $n \geqslant 1$. Hence the expectation of the number of jumps $N_\varepsilon(T)$ of $\boldsymbol{Y}_\varepsilon$ in $[0, T]$ is at most $1/(c_0\varepsilon\theta)$ since

$$E[N_\varepsilon(T)] = \sum_{n=1}^{\infty} P(T_n < T) \leqslant \sum_{n=1}^{\infty} (1 - c_0\varepsilon\theta)^{n-1} = \frac{1}{c_0\varepsilon\theta}.$$

The above method has been used to solve locally the partial differential equation

$$\frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} + 3\frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} = -16 \qquad (x_1, x_2) \in D = (0, 1) \times (0, 1)$$

with the mixed boundary conditions $u(x_1, 1) = -3.5349x_1^2 + 0.5161x_1 + 3.0441$, $u(1, x_2) = 0$, $\partial u/\partial x_1 = 0$ on $\{0\} \times (0, 1)$, and $\partial u/\partial x_2 = 0$ on $(0, 1) \times \{0\}$. The $\mathbb{R}^2$-valued diffusion process $\boldsymbol{X} = (B_1, \sqrt{3}B_2)$ is used to find u at arbitrary points in D, where B_1 and B_2 are independent Brownian motions. The results by the local solution with $\varepsilon = 0.01$ and $n_s = 5000$ samples of $\boldsymbol{X}$ generated with a time step $\Delta t = 0.005$ are within 3% of the finite difference solution at the tested points in D. The local solutions at points in D relatively far from Neumann boundaries do not seem to be sensitive to the particular value of ε, but they depend on ε at points close to Neumann boundaries. For example, the difference between the local and finite difference solutions at (0.25, 0.25) is 10.75%, 6.35% and 1.10% for $\varepsilon = 0.1$, 0.05 and 0.01, respectively, for $\Delta t = 0.005$ and $n_s = 5000$.

4. Comments

Methods based on properties of diffusion processes and the Itô formula for continuous semimartingales have been applied successfully to solve locally a broad class of partial differential equations with Dirichlet boundary conditions. The extension of these methods to the local solution of partial differential equations with mixed boundary conditions poses significant difficulties since the solution is not known at all boundary points. This paper has developed a method for solving locally a class of partial differential equations with mixed boundary conditions. The method is based on a heuristic idea according to which the diffusion process used for solving a partial differential equation is stopped when it reaches a Neumann boundary and then restarted inside the domain of definition of the equation at a point depending on the Neumann conditions. Properties of diffusion processes, submartingales, and stopping times, and the Itô formula for semimartingales are used to prove the validity of the proposed local method. The proposed method can be applied to solve locally any one-dimensional

mixed boundary value problem. For higher dimensional mixed boundary value problems the method is valid only for a restricted class of problems that admit regular solutions.

Two simple examples have been presented to illustrate the local method in the paper. Numerical results show that the method is accurate. The expectations in the expression of the local method have been estimated by Monte Carlo simulation. The numerical algorithms for solution are simple to program and ideal for parallel computation.

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